On the Convergence of Quadrature Formulas Connected with Multipoint Padé-Type Approximation

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Let $I(F) = \int_{-1}^{1} F(x)\omega(x)\,dx$, where ω is a complex valued integrable function. We consider quadrature formulas for $I(F)$ which are exact with respect to rational functions with prescribed poles contained in $\overline{\mathbb{C}} \setminus [-1, 1]$. Their rate of convergence is studied.

1. INTRODUCTION

Let μ be a finite complex Borel measure, supported on $[-1, 1]$, that is, $S_\mu = \operatorname{supp}(\mu) \subset [-1, 1]$ and $\int d|\mu| = k < +\infty$, where $|\mu|$ denotes the total variation measure associated with μ. Occasionally, we restrict our attention to measures of the form $d\mu(x) = \omega(x)\,dx$, where ω is a complex valued function. In this case, $|d\mu(x)| = |\omega(x)|\,dx$. Let $\hat{\mu}(z)$ denote the

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corresponding Markov function

$$\hat{\mu}(z) = \int_{-1}^{1} \frac{d\mu(x)}{z - x} = \int \frac{d\mu}{z - x}. \tag{1.1}$$

In this paper we shall be concerned with the approximation of integrals of the form

$$I_\mu(F) = \int F(x)\, d\mu(x) \qquad \text{or} \qquad I_\omega(F) = \int F(x)\,\omega(x)\, dx, \tag{1.2}$$

by means of interpolatory quadrature formulas

$$I_n(F) = \sum_{j=1}^{N} \sum_{k=0}^{M_j - 1} A_{k,j}^{n} F^{(k)}(x_{n,j}), \tag{1.3}$$

with $n = M_1 + \cdots + M_N$ which are exact for certain types of rational functions with prescribed poles.

Motivated by a paper of Nuttall and Wherry [2], we treated the same problem in [4] but in the context of quadrature formulas which are exact for polynomials. There, we considered not only interpolatory-type quadratures but also Jacobi-type and compared their rates of convergence. Here, we limit ourselves to the interpolatory case for reasons which we shall explain in Remark 3 of Section 3.

The general outline of the paper is the following. In Section 2, the connection between interpolatory-type quadrature formulas which are exact for rational functions and multipoint Padé-type approximants of Markov functions is established. In Section 3, using the rate of convergence of multipoint Padé-type approximants to Markov functions, we estimate the rate of convergence of quadrature rules of type (1.3) when F is holomorphic on a neighborhood of S_μ. In Section 4, we study the case when F is only defined on $[-1, 1]$ and belongs to the Lipschitz class $\mathrm{Lip}(\alpha)$, $\alpha > \frac{1}{2}$. Some concluding remarks are also included in this section. Finally, numerical examples are displayed in Section 5.

2. PRELIMINARY RESULTS

In the following, Π_n will denote the space of polynomials of degree at most equal to n, and Π the space of all polynomials. Let $\alpha_n = \{\alpha_{n,j} : j = 1, \dots, n\}$ and $\beta_n = \{\beta_{n,j} : j = 1, \dots, n\}$ be two sets of points such that $\alpha_n \subset \overline{\mathbb{C}} \setminus [-1, 1]$ and $\beta_n \subset \mathbb{C} \setminus \alpha_n$, $n \in \mathbb{N}$, is fixed (as usual, $\overline{\mathbb{C}}$ represents the extended complex plane). Set

$$w_n^{\alpha}(z) = \prod_{j=1}^{n} \left(1 - z\alpha_{n,j}^{-1}\right) \tag{2.1}$$

$$w_n^{\beta}(z) = \prod_{j=1}^{n} (z - \beta_{n,j}), \tag{2.2}$$

where the factors in (2.1) corresponding to $\alpha_{n,j} = \infty$ are considered equal to one.

It is easy to verify that there exists a unique polynomial $Q_{n-1} \in \Pi_{n-1}$ such that

$$\left(\frac{w_n^{\beta}\hat{\mu} - Q_{n-1}}{w_n^{\alpha}}\right)(z) = O\left(\frac{1}{z}\right) \in \mathscr{H}\left(\overline{\mathbb{C}} \setminus [-1,1]\right), \tag{2.3}$$

where $O(z^{-1}) \to 0$ when $(|z| \to \infty)$. If $\deg(w_n^{\alpha}) = n$, then the condition $O(z^{-1})$ is automatically fulfilled. When $\deg(w_n^{\alpha}) = n - \lambda_n$ $(1 \le \lambda_n \le n)$, this requirement imposes λ_n conditions of interpolation at infinity. On the other hand, the construction of Q_{n-1} involves the interpolation of $(w_n^{\beta}\hat{\mu})$ at the zeros of w_n^{α}.

The rational function

$$\frac{Q_{n-1}(z)}{w_n^{\beta}(z)} = (n - 1/n)_{\hat{\mu}}(z)$$

is called an $(n - 1/n)$ multipoint Padé-type approximant (MPTA) of $\hat{\mu}(z)$ relative to (α_n, β_n). For a general class of meromorphic functions on $\overline{\mathbb{C}} \setminus [-1,1]$ (including Cauchy transforms of complex measures on $[-1,1]$), $w_n^{\alpha} \equiv 1$, $n \in \mathbb{N}$, and $\{w_n^{\beta}\}$ any sequence of classical orthogonal polynomials, the convergence of such MPTA was studied by Gonchar (see Theorem 2 in [5]).

Let us see how one obtains quadrature formulas (1.3) which are exact for rational functions of the form

$$R(x) = \frac{P(x)}{w_n^{\alpha}(x)}, \qquad P \in \Pi_{n-1}.$$

Fix any piecewise smooth curve Γ such that $[-1,1] \cup \beta_n \subset \operatorname{Int}(\Gamma)$ and $\alpha_n \subset \operatorname{Ext}(\Gamma)$. From (2.3), one has

$$\left(\frac{\hat{\mu}}{w_n^{\alpha}} - \frac{Q_{n-1}}{w_n^{\alpha} w_n^{\beta}}\right)(z) = O\left((z^{-1})^{1+n}\right) \in \mathscr{H}(\operatorname{Ext}(\Gamma)).$$

Therefore, using Cauchy's Theorem on Ext(Γ), one has

$$\int_\Gamma P(z)\left(\frac{\hat{\mu}}{w_n^\alpha} - \frac{Q_{n-1}}{w_n^\alpha w_n^\beta}\right)(z)\,dz = 0, \qquad \forall P \in \Pi_{n-1},$$

or what is the same,

$$\int_\Gamma \frac{P(z)\hat{\mu}(z)}{w_n^\alpha(z)}\,dz = \int_\Gamma \frac{P(z)Q_{n-1}(z)}{w_n^\alpha(z)w_n^\beta(z)}\,dz. \tag{2.4}$$

Fubini's Theorem implies that

$$\int_\Gamma \frac{P(z)\hat{\mu}(z)}{w_n^\alpha(z)}\,dz = \int\int_\Gamma \frac{P(z)}{w_n^\alpha(z)}\frac{dz}{z-x}\,d\mu(x)$$

$$= 2\pi i\int \frac{P(x)}{w_n^\alpha(x)}\,d\mu(x). \tag{2.5}$$

Let us consider the simple fraction expansion of Q_{n-1}/w_n^β. Assume that $w_n^\beta(z) = \prod_{j+1}^{N}(z-\beta_{n,j})^{M_j}$, where $\beta_{n,j_1} \neq \beta_{n,j_2}$ for $j_1 \neq j_2$. Then

$$\frac{Q_{n-1}(z)}{w_n^\beta(z)} = \sum_{j=1}^{N}\sum_{k=0}^{M_j-1}\frac{k!A_{k,j}^n}{\left(z-\beta_{n,j}\right)^{k+1}} \tag{2.6}$$

and, using Cauchy's integral formula for the derivatives, one obtains

$$\int_\Gamma \frac{P(z)}{w_n^\alpha(z)}\frac{Q_{n-1}(z)}{w_n^\beta(z)}\,dz = \sum_{j=1}^{N}\sum_{k=0}^{M_j-1}k!A_{k,j}^n\int_\Gamma \frac{P(z)}{w_n^\alpha(z)}\frac{dz}{\left(z-\beta_{n,j}\right)^{k+1}}$$

$$= 2\pi i\sum_{j=1}^{N}\sum_{k=0}^{M_j-1}A_{k,j}^n\left(\frac{P}{w_n^\alpha}\right)^{(k)}(\beta_{n,j}). \tag{2.7}$$

From (2.4), (2.5), and (2.7), follows

LEMMA 1. *Under the conditions above, we have*

$$\int \frac{P(x)}{w_n^\alpha(x)}\,d\mu(x) = \sum_{j=1}^{N}\sum_{k=0}^{M_j-1}A_{k,j}^n\left(\frac{P}{w_n^\alpha}\right)^{(k)}(\beta_{n,j}), \qquad P \in \Pi_{n-1}, \tag{2.8}$$

where $A_{k,j}^n$ is given by (2.6). *In particular, if all the zeros of w_n^β are simple, then*

$$\int \frac{P(x)}{w_n^\alpha(x)}\,d\mu(x) = \sum_{j=1}^{n}A_j^n\frac{P(\beta_{n,j})}{w_n^\alpha(\beta_{n,j})},$$

with

$$A_j^n = \frac{Q_{n-1}(\beta_{n,j})}{(\omega_n^\beta)'(\beta_{n,j})}.$$

Remark 1. The following converse statement to Lemma 1 is not hard to obtain. Assume that you have a quadrature formula of the form (1.3) which is exact for all rational functions of the form P/w_n^α, $P \in \Pi_{n-1}$, where w_n^α is a given polynomial of degree at most n, and whose zeros lie in $\mathbb{C} \setminus [-1,1]$. Set $w_n^\beta(z) = \prod_{j=1}^N (z-\beta_{n,j})^{M_j}$, where $\beta_{n,j}$ and M_j, $j = 1 \dots n$, are taken from (1.3). Assume that w_n^α and w_n^β have no common zeros. Then, defining Q_{n-1} by formula (2.6), where the coefficients $A_{k,j}^n$ are taken from (1.3), you have that the relation (2.3) holds. Therefore, there is a one-to-one correspondence between quadrature formulas of the form (1.3) and MPTA of $\hat{\mu}$ under the conditions that the zeros of w_n^α lie in $\overline{\mathbb{C}}/[-1,1]$ and that w_n^α and w_n^β are relatively prime.

Now, let us find an expression linking the error in the quadrature formula and the error in the MPTA.

LEMMA 2. *Let F be an analytic function on a domain V and let Γ be an arbitrary piecewise smooth Jordan curve such that* $[-1,1] \cup \beta_n \subset \operatorname{Int}(\Gamma)$, $\overline{\operatorname{Int}(\Gamma)} \subset V$, *and* $\alpha_n \subset \operatorname{Ext}(\Gamma)$. *Then*

$$E_n(F) = I_\mu(F) - I_n(F) = \frac{1}{2\pi i}\int_\Gamma F(z)\left(\hat{\mu}(z) - \frac{Q_{n-1}(z)}{w_n^\beta(z)}\right) dz. \quad (2.9)$$

Proof. Fubini's Theorem and Cauchy's integral formula imply that

$$\frac{1}{2\pi i}\int_\Gamma F(z)\hat{\mu}(z)\,dz = \int \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-x}\,dz\,d\mu(x) = \int F(x)\,d\mu(x).$$

On the other hand, from (2.6), one finds that

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma F(z)\frac{Q_{n-1}(z)}{w_n^\beta(z)}\,dz &= \sum_{j=1}^{N}\sum_{k=0}^{M_j-1}\frac{k!A_{k,j}^n}{2\pi i}\int_\Gamma \frac{F(z)}{(z-\beta_{n,j})^{k+1}}\,dz\\ &= I_n(F).\end{aligned}$$

These two formulas give (2.9). ∎

Note that (2.8) is a particular case of (2.9). In fact, taking $F = P/w_n^\alpha$, $P \in \Pi_{n-1}$, one would have that $(P/w_n^\alpha)(\hat{\mu} - Q_{n-1}/w_n^\beta)$ is holomorphic in Ext(Γ), with a zero of multiplicity at least two at infinity. Therefore, the right-hand side of (2.9) equals zero.

To conclude this section, we give an integral expression for the error in the MPTA.

LEMMA 3. *Let α_n and β_n be as in Lemma 1. Then*

$$\hat{\mu}(z) - \frac{Q_{n-1}(z)}{w_n^\beta(z)} = \frac{w_n^\alpha(z)}{w_n^\beta(z)} \int \frac{w_n^\beta(x)\, d\mu(x)}{w_n^\alpha(x)(z-x)} \tag{2.10}$$

and

$$Q_{n-1}(z) = \int \left(\frac{w_n^\alpha(x) w_n^\beta(z) - w_n^\alpha(z) w_n^\beta(x)}{w_n^\alpha(x)} \right) \frac{d\mu(x)}{z-x}; \tag{2.11}$$

in particular, if $\beta_{n,j}$ is a simple zero of w_n^β, then

$$A_j^n = \frac{w_n^\alpha(\beta_{n,j})}{(w_n^\beta)'(\beta_{n,j})} \int \frac{w_n^\beta(x)}{w_n^\alpha(x)(x-\beta_{n,j})}\, d\mu(x). \tag{2.12}$$

Proof. Let Γ be a piecewise smooth curve separating $[-1,1] \cup \beta_n$ from α_n. Let $z \in \mathrm{Ext}(\Gamma)$. Since

$$\frac{w_n^\beta \hat{\mu} - Q_{n-1}}{w_n^\alpha} \in \mathscr{H}\left(\overline{\mathrm{Ext}(\Gamma)}\right)$$

and has a zero of multiplicity at least one at $z = \infty$, we can write, by using Fubini's Theorem and Cauchy's integral formula on Ext(Γ),

$$\begin{aligned} \left(\frac{w_n^\beta \hat{\mu} - Q_{n-1}}{w_n^\alpha} \right)(z) &= -\frac{1}{2\pi i} \int_\Gamma \left(\frac{w_n^\beta \hat{\mu} - Q_{n-1}}{w_n^\alpha} \right)(\xi) \frac{d\xi}{\xi - z} \\ &= \int \frac{w_n^\beta(x)\, d\mu(x)}{w_n^\alpha(x)(z-x)}. \end{aligned}$$

This formula immediately yields (2.10). Then (2.11) follows (2.10) by obvious algebraic transformations. Finally, (2.12) is obtained from (2.11) using the trivial fact that if $\beta_{n,j}$ is simple then

$$A_j^n = \operatorname{Res}\left(\frac{Q_{n-1}}{w_n^\beta}, \beta_{n,j}\right) = \lim_{z \to \beta_{n,j}} (z - \beta_{n,j}) \frac{Q_{n-1}(z)}{w_n^\beta(z)}. \quad \blacksquare$$

3. CONVERGENCE FOR ANALYTIC FUNCTIONS

In the following, $\alpha = \{\alpha_n\}$, $n \in \mathbb{N}$, denotes a triangular table of points compactly contained in $\overline{\mathbb{C}} \setminus [-1, 1]$.

Let $\beta = \{\beta_n\}$, $n \in \mathbb{N}$, be a triangular table of points such that $\beta \subset \mathbb{C} \setminus \alpha$. Moreover, the set β' of limit points of β is contained in $[-1, 1]$. A point β_0 belongs to β' is there exists a subsequence $\{\beta_{n, j(n)}\}$, $n \in \mathbb{N}$, such that $\beta_{n, j(n)} \to \beta_0$ as $n \to \infty$.

We wish to make proper selections of the tables α and β to obtain good estimates for the rate of convergence of the sequence of MPTA Q_{n-1}/w_n^β, $n \in \mathbb{N}$, to $\hat{\mu}$. Using (2.9), we deduce fine bounds for the rate of convergence of the corresponding sequence of interpolatory quadrature formulas when F is holomorphic on a neighborhood of $[-1, 1]$.

To avoid the annoying effect of the difference in which we wrote the polynomials w_n^α and w_n^β (see (2.1), (2.2)), we will assume in the following that $\deg(w_n^\alpha) = n$ for each n and take

$$w_n^\alpha(z) = \prod_{j=1}^{n} (z - \alpha_{n,j}). \tag{3.1}$$

This may be done, without loss of generality, by taking a convenient bilinear transformation which maps $[-1, 1]$ onto itself and maps infinity into some finite point, but takes no original $\alpha_{n,j}$ into infinity (this is possible since, by assumption, α is compactly contained in $\overline{\mathbb{C}} \setminus [-1, 1]$).

Let P be a polynomial of degree n. We associate to it the atomic measure

$$\nu_n(P) = \frac{1}{n} \sum_{P(\zeta)=0} \delta_\zeta,$$

where δ_ζ denotes the Dirac measure supported at point ζ. The corresponding measures for w_n^α and w_n^β we denote by ν_n^α and ν_n^β, respectively. The logarithmic potential of a measure ν is given by

$$V_\nu(z) = \int \log \frac{1}{|z - \zeta|} \, d\nu(\zeta).$$

In particular,

$$\left|w_n^{\beta}(z)\right|^{1/n} = \exp\left\{-V_{\nu_n^{\beta}}(z)\right\}, \qquad \left|w_n^{\alpha}(z)\right|^{1/n} = \exp\left\{-V_{\nu_n^{\alpha}}(z)\right\}. \quad (3.2)$$

Assume that there exist measures ν^{α} and ν^{β} such that

$$\nu_n^{\alpha} \underset{n\in\mathbb{N}}{\overset{*}{\rightarrow}} \nu^{\alpha} \qquad \text{and} \qquad \nu_n^{\beta} \underset{n\in\mathbb{N}}{\overset{*}{\rightarrow}} \nu^{\beta} \quad (3.3)$$

in the weak star topology of the space of measures. This implies that (see [9])

$$\lim_{n\to\infty} \left|w_n^{\alpha}(z)\right|^{1/n} = \exp\left\{-V_{\nu^{\alpha}}(z)\right\}, \qquad z \in \mathbb{C}\setminus \operatorname{supp}(\nu^{\alpha}), \quad (3.4)$$

$$\limsup_{n\to\infty} \left|w_n^{\alpha}(z)\right|^{1/n} \le \exp\left\{-V_{\nu^{\alpha}}(z)\right\}, \qquad z \in \mathbb{C}, \quad (3.5)$$

$$\lim_{n\to\infty} \left|w_n^{\beta}(z)\right|^{1/n} = \exp\left\{-V_{\nu^{\beta}}(z)\right\}, \qquad z \in \mathbb{C}\setminus \operatorname{supp}(\nu^{\beta}), \quad (3.6)$$

$$\limsup_{n\to\infty} \left|w_n^{\beta}(z)\right|^{1/n} \le \exp\left\{-V_{\nu^{\beta}}(z)\right\}, \qquad z \in \mathbb{C}. \quad (3.7)$$

Convergence in (3.4)–(3.7) is uniform on each compact subset of the indicated regions. Since $\beta' \subset [-1,1]$ then, obviously, $\operatorname{supp}(\nu^{\beta}) \subset [-1,1]$ and (3.6) takes place, in particular, in $\mathbb{C}\setminus[-1,1]$. For analogous reasons, since α is compactly contained in $\mathbb{C}\setminus[-1,1]$, then $\operatorname{supp}(\nu^{\alpha}) \subset \mathbb{C}\setminus[-1,1]$ and (3.4) holds on $[-1,1]$.

From (2.10) and (3.4)–(3.7) one finds that

$$\begin{aligned}\limsup_{n\to\infty}\left|\left(\hat{\mu} - \frac{Q_{n-1}}{w_n^{\beta}}\right)(z)\right|^{1/n} &\le \frac{\limsup_{n\to\infty}\left|w_n^{\alpha}(z)/w_n^{\beta}(z)\right|^{1/n}}{\liminf_{n\to\infty}\left\{\min_{x\in[-1,1]}\left|w_n^{\alpha}(x)/w_n^{\beta}(x)\right|\right\}^{1/n}} \\ &\le \exp\left\{V_{\nu^{\beta}-\nu^{\alpha}}(z) - \min_{x\in[-1,1]} V_{\nu^{\beta}-\nu^{\alpha}}(x)\right\},\end{aligned} \quad (3.8)$$

where the limit in (3.8) is uniform on compact subsets of $\mathbb{C}\setminus[-1,1]$.

Now, $V_{\nu^{\beta}-\nu^{\alpha}}$ is subharmonic in $\overline{\mathbb{C}}\setminus[-1,1]$ and, therefore,

$$V_{\nu^{\beta}-\nu^{\alpha}}(z) < \max_{x\in[-1,1]} V_{\nu^{\beta}-\nu^{\alpha}}(x), \qquad z \in \overline{\mathbb{C}}\setminus[-1,1]. \quad (3.9)$$

Relations (3.8) and (3.9) clearly indicate that it is convenient to have

$$\min_{x \in [-1,1]} V_{\nu^\beta - \nu^\alpha}(x) = \max_{x \in [-1,1]} V_{\nu^\beta - \nu^\alpha}(x),$$

or what is the same, $V_{\nu^\beta - \nu^\alpha}$ equal to a constant on $[-1,1]$. This will ensure convergence of the sequence of MPTA on all $\overline{\mathbb{C}} \setminus [-1,1]$.

This is possible if $\nu^\beta = \tilde{\nu}^\alpha$ is the balayage of ν^α from $\mathbb{C} \setminus [-1,1]$ on $[-1,1]$. That is, ν^β must be the equilibrium distribution on $[-1,1]$ in the presence of the exterior field $-V_{\nu^\alpha}$ (see [10, 9, 6]). Thus, we have

THEOREM 1. *Let α and β be two triangular tables as described above. Assume that* (3.3) *takes place and $\nu^\beta = \tilde{\nu}^\alpha$ is the equilibrium measure on $[-1,1]$ in the presence of the exterior field $-V_{\nu^\alpha}$. Then*

$$\limsup_{n \to \infty} \left| \left(\hat{\mu} - \frac{Q_{n-1}}{w_n^\beta} \right)(z) \right|^{1/n} \leq \exp\{ V_{\tilde{\nu}^\alpha - \nu^\alpha}(z) - C \} < 1, \qquad z \in \overline{\mathbb{C}} \setminus [-1,1], \tag{3.10}$$

where

$$C = V_{\tilde{\nu}^\alpha - \nu^\alpha}(x), \qquad x \in [-1,1],$$

is the equilibrium constant. The limit in (3.10) *is uniform on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.*

The proof is immediate from (3.8) and the reasonings above. Alternative formulas for the right-hand side of (3.10) are well known in terms of Green's potential. Let $g_{[-1,1]}(z,t)$ denote the Green's function for the region $\overline{\mathbb{C}} \setminus [-1,1]$ with singularity at point $t \in \overline{\mathbb{C}} \setminus [-1,1]$. Then (see [9])

$$C = \int g_{[-1,1]}(t, \infty)\, d\nu^\alpha(t)$$

and

$$V_{\tilde{\nu}^\alpha - \nu^\alpha}(z) = C - \int g_{[-1,1]}(z,t)\, d\nu^\alpha(t), \qquad z \in \overline{\mathbb{C}} \setminus [-1,1].$$

Therefore,

$$C - V_{\tilde{\nu}^\alpha - \nu^\alpha}(z) = \int g_{[-1,1]}(z,t)\, d\nu^\alpha(t) = G^\alpha_{[-1,1]}(z),$$

and thus,

$$\limsup_{n \to \infty} \left| \left(\hat{\mu} - \frac{Q_{n-1}}{w_n^\beta} \right)(z) \right|^{1/n} \leq \exp\{ -G^\alpha_{[-1,1]}(z) \}. \tag{3.11}$$

Then, for the error $E_n(F)$ of the quadrature formula given by (2.9), we have

THEOREM 2. *Let F denote an analytic function on a neighborhood V of $[-1,1]$ and assume that the tables α, β satisfy the conditions of Theorem 1. Then*

$$\limsup_{n\to\infty} |E_n(F)|^{1/n} \le \exp\{-\tau\} < 1, \tag{3.12}$$

where $\tau = \sup\{\rho \colon \Gamma_\rho \subset V\}$ and $\Gamma_\rho = \{z \colon G^{\alpha}_{[-1,1]}(z) = \rho\}$. Here, τ is related to the largest equipotential curve contained in V.

Proof. Let $\tau > 0$ be such that $\Gamma_\rho \subset V$. For all sufficiently large n all the points in β_n lie in $V_\rho = \{z \colon G^{\alpha}_{[-1,1]}(z) < \rho\} \subset V$ and formula (2.9) makes sense with $\Gamma = \Gamma_\rho$. Hence

$$\begin{aligned} |E_n(F)| &\le \frac{1}{2\pi}\int_{\Gamma_\rho} |F(z)| \left| \hat{\mu}(z) - \frac{Q_{n-1}(z)}{w_n^{\beta}(z)} \right| |dz| \\ &\le \frac{\mathscr{L}_\rho M_\rho}{2\pi} \sup_{z\in\Gamma_\rho} \left| \hat{\mu}(z) - \frac{Q_{n-1}(z)}{w_n^{\beta}(z)} \right|, \end{aligned} \tag{3.13}$$

where $\mathscr{L}_\rho$ denotes the length of the curve Γ_ρ and $M_\rho = \sup_{z\in\Gamma_\rho}|F(z)|$. For just one value of ρ, the length of Γ_ρ may be infinite (that corresponding to the value passing through infinity) and one excludes that value in case that it is possible.

From (3.13) and (3.11) (or (3.10)) one obtains

$$\limsup_{n\to\infty} |E_n(F)|^{1/n} \le \exp\{-\rho\}.$$

Since this is true for each ρ such that $\Gamma_\rho \subset V$, one arrives at (3.12) making ρ tend to τ. ∎

Remark 2. Before ending this section we wish to point out how it is possible to construct a table of points β with the property required by Theorem 1, given a table α for which

$$\nu_n^{\alpha} \underset{n\in\mathbb{N}}{\overset{*}{\to}} \nu^{\alpha}. \tag{3.14}$$

Such constructions are known (see e.g. [6]).

Take any positive Borel measure σ supported on $[-1,1]$ and such that $\sigma' > 0$ a.e. on $[-1,1]$. Take w_n^{β} as the nth orthogonal polynomial with respect to the varying measure $d\sigma_n = d\sigma/|w_n^\alpha|^2$. If α satisfies (3.14) then

$$\nu_n^\beta \overset{*}{\underset{n\in\mathbb{N}}{\to}} \tilde{\nu}^\alpha.$$

Remark 3. Results analogous to those of Theorems 1 and 2 may be obtained in connection with multipoint Padé approximants of Markov functions of complex measures and the Jacobi-type quadrature formulas they generate. A serious difficulty arises. Orthogonal polynomials with respect to complex measures need not have degree n and, moreover, if they do, their zeros may be anywhere in the complex plain. In [4], where we consider the special case $w_n^\alpha \equiv 1$, $n \in \mathbb{N}$, we could rely on known results concerning the asymptotic behavior of polynomials orthogonal with respect to fixed complex measures and their zeros to overcome these difficulties. Now, we are dealing with varying complex measures $d\mu(x)/w_n^\alpha(x)$ and this is no longer so. We shall treat such questions in a separate paper.

4. CONVERGENCE FOR CONTINUOUS FUNCTIONS

In this section F is only defined on $[-1,1]$. Therefore, we must take β so that all its points lie within this interval. Since we will no longer work with potentials, we return to the expression (3.1) for the polynomials w_n^α. As in Section 3, we consider the case where the table α is compactly contained in $\overline{\mathbb{C}} \setminus [-1,1]$. We will no longer assume any nth root asymptotic behavior for the sequence $\{w_n^\alpha\}$; but, given α, we will make a convenient choice for the polynomials w_n^β.

Let ω be any complex-valued measurable function on $[-1,1]$ such that

$$\int |\omega(x)|\, dx < +\infty. \tag{4.1}$$

Let $h(x)$ be a weight function defined on $[-1,1]$, such that $h(x) > 0$ a.e. on $[-1,1]$ and

$$\int h(x)\, dx < \infty. \tag{4.2}$$

Assume that

$$\int \frac{|\omega(x)|^2}{h(x)}\, dx = C_1 < +\infty. \tag{4.3}$$

Given α, take w_n^β as the nth orthogonal polynomial with respect to the measure $h(x)\,dx/|w_n^\alpha(x)|^2$. Certainly, for each $n \in \mathbb{N}$, all the zeros of w_n^β are simple and lie on $[-1, 1]$. In particular, for $d\mu(x) = \omega(x)\,dx$, according to (2.12) we have

$$A_j^n = \frac{w_n^\alpha(\beta_{n,j})}{(w_n^\beta)'(\beta_{n,j})} \int \frac{w_n^\beta(x)}{w_n^\alpha(x)} \frac{\omega(x)\,dx}{(x - \beta_{n,j})} = \frac{Q_{n-1}(\beta_{n,j})}{(w_n^\beta)'(\beta_{n,j})}. \tag{4.4}$$

From the Gauss–Jacobi quadrature formula we know that (for a similar treatment, see [7])

$$\int P(x) \frac{h(x)\,dx}{|w_n^\alpha(x)|^2} = \sum_{j=1}^n \lambda_j^n P(\beta_{n,j}), \qquad P \in \Pi_{2n-1}, \tag{4.5}$$

and

$$\lambda_j^n = \frac{1}{\left((w_n^\beta)'(\beta_{n,j})\right)^2} \int \frac{\left(w_n^\beta(x)\right)^2}{|w_n^\alpha(x)|^2} \frac{h(x)\,dx}{(x - \beta_{n,j})^2}.$$

It is convenient to write these formulas in a more symmetric form by multiplying and dividing each term in the right-hand side of (4.5) by $|w_n^\alpha(\beta_{n,j})|^2$, respectively. Thus, we obtain

$$\int \frac{P(x)}{|w_n^\alpha(x)|^2} h(x)\,dx = \sum_{j=1}^n \tilde{\lambda}_j^n \frac{P(\beta_{n,j})}{|w_n^\alpha(\beta_{n,j})|^2}, \qquad P \in \Pi_{2n-1}, \tag{4.6}$$

with

$$\tilde{\lambda}_j^n = \left| \frac{w_n^\alpha(\beta_{n,j})}{(w_n^\beta)'(\beta_{n,j})} \right|^2 \int \left| \frac{w_n^\beta(x)}{w_n^\alpha(x)} \right|^2 \frac{h(x)\,dx}{(x - \beta_{n,j})^2}. \tag{4.7}$$

Let us try to establish some connection between the numbers A_j^n in (4.4) and the $\tilde{\lambda}_j^n$ in (4.7).

LEMMA 4. *Assume that h and ω satisfy* (4.1)–(4.3). *Take α and β as indicated above. Then*

$$|A_j^n| \le \sqrt{C_1 \tilde{\lambda}_j^n} \tag{4.8}$$

and

$$\sum_{j=1}^{n} |A_j^n| \le C_2 \sqrt{n}\,, \tag{4.9}$$

where C_2 is an absolute constant.

Proof. Multiplying and dividing the integrand in (4.4) by $\sqrt{h(x)}$ and using the Cauchy–Schwarz inequality and (4.7), one has

$$\begin{aligned} |A_j^n| &\le \left| \frac{w_n^{\alpha}(\beta_{n,j})}{(w_n^{\beta})'(\beta_{n,j})} \right| \left[\int \left| \frac{w_n^{\beta}(x)}{w_n^{\alpha}(x)} \right|^2 \frac{h(x)\,dx}{(x-\beta_{n,j})^2} \right]^{1/2} \left[\int \frac{|\omega(x)|^2}{h(x)}\,dx \right]^{1/2} \\ &= \sqrt{C_1 \tilde{\lambda}_j^n}\,. \end{aligned}$$

This gives us (4.8).

It is a known fact (see [7]) that

$$\lim_{n \to \infty} \sum_{j=1}^{n} \tilde{\lambda}_j^n f(\beta_{n,j}) = \int f(x) h(x)\,dx,$$

where f is any continuous function on $[-1,1]$. In particular, taking $f \equiv 1$ we have that there exists a constant C_3 such that

$$\sum_{j=1}^{n} \tilde{\lambda}_j^n \le C_3.$$

Therefore, using (4.8) and the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned} \sum_{j=1}^{n} |A_j^n| &\le \sqrt{C_1} \sum_{j=1}^{n} \sqrt{\tilde{\lambda}_j^n} \le \sqrt{C_1}\sqrt{n} \sqrt{\sum_{j=1}^{n} \tilde{\lambda}_j^n} \\ &= \sqrt{C_1 C_3}\sqrt{n}\,, \end{aligned}$$

which proves (4.9). ∎

Now we are ready for

THEOREM 3. *Under the assumptions of Lemma* 4,

$$|I(F) - I_n(F)| \le C_4 \sqrt{n}\, \rho_{n-1}(F), \tag{4.10}$$

where C_4 is an absolute constant, F is an arbitrary continuous function on $[-1,1]$, *and*

$$\rho_{n-1}(F) = \inf_{P \in \Pi_{n-1}} \left\| F - \frac{P}{w_n^{\alpha}} \right\|_{\infty}. \tag{4.11}$$

Proof. Let $P_{n-1} \in \Pi_{n-1}$ be such that

$$\left\| F - \frac{P_{n-1}}{w_n^\alpha} \right\|_\infty = \rho_{n-1}(F).$$

Then, according to (2.8), (4.1), and (4.9),

$$\begin{aligned} |I(F) - I_n(F)| &= \left| I\left(F - \frac{P_{n-1}}{w_n^\alpha} \right) + I_n\left(\frac{P_{n-1}}{w_n^\alpha} - F \right) \right| \\ &\leq \rho_{n-1}(F) \left(\int |\omega(x)|\, dx + C_2 \sqrt{n} \right). \end{aligned}$$

This settles (4.10). ∎

In order to give sufficient conditions on F so that the right-hand side of (4.10) tends to zero, we must obtain estimates of $\rho_{n-1}(F)$ in terms of n for continuous F. Such estimates are well known in the case that F is approximated by polynomials. Bernstein's Theorem states (see [11]) that in that case

$$\rho_{n-1}(F) \leq C_5 \, \omega\left(F, \frac{1}{n} \right),$$

where C_5 is an absolute constant (independent of F and n) and $\omega(F, \delta) = \sup_{\substack{|x-y| \leq \delta \\ x, y \in [-1,1]}} |F(x) - F(y)|$ *denotes the modulus of continuity of* F *on* $[-1, 1]$.

Given α_0, a fixed point contained in $\mathbb{R} \setminus [-1, 1]$, a bilinear transformation yields that

$$\rho_{n-1}(F) \leq C_6 \, \omega\left(F, \frac{1}{n} \right),$$

where C_6 is in general a different absolute constant and ρ_{n-1} denotes the best approximation of F by means of rational functions of order $n - 1$ whose only poles lie at the point α_0.

It seems natural to expect that if we look for the best approximation of F by means of rational functions with prescribed poles, the same type of estimate for ρ_{n-1} should hold as long as the poles of the approximating rational functions do not approach the interval where F is being approximated as $n \to \infty$ (or if they do approach the interval, this is done "not too fast"). We thought such a result should be known but found no reference; therefore, for completeness, we include a proof. We limit ourselves to the case when the table of points α (which determines the fixed points) is compactly contained in $\mathbb{C} \setminus [-1, 1]$ and (3.4) takes place. We have

THEOREM 4. *Let* α *be compactly contained in* $\mathbb{C} \setminus [-1, 1]$, *such that* (3.4) *takes place, and* F *be a non-constant continuous function on* $[-1, 1]$.

Then, there exist absolute constants C_7 and C_8 such that, for sufficiently large n,

$$\rho_{n-1}(F) \leq [C_7 + C_8 C(F)]\, \omega\left(F, \frac{1}{n}\right), \tag{4.12}$$

where $\rho_{n-1}(F)$ is as in (4.11) *and*

$$C(F) = \frac{\|F\|_\infty}{\max_{x \in [-1,1]} F(x) - \min_{x \in [-1,1]} F(x)}.$$

Proof. Let p_m^* be the polynomials of degree m of best uniform approximation to F on $[-1, 1]$. Since $\lim_m p_m^* = F$ uniformly on $[-1, 1]$, then for all sufficiently large m, $\|p_m^*\|_\infty \leq 2\|F\|_\infty$. In the following, we only consider such m. From the Bernstein–Walsh formula (see [12, p. 77]), we obtain that

$$|p_m^*(z)| \leq 2\|F\|_\infty \exp\{m g_{[-1,1]}(z, \infty)\}. \tag{4.13}$$

As above, take a table of points β contained in $[-1, 1]$ such hat

$$\nu_n^\beta \underset{n \in \mathbb{N}}{\overset{*}{\rightarrow}} \tilde{\nu}^\alpha$$

where $\tilde{\nu}^\alpha$ is the equilibrium measure on $[-1, 1]$ in the presence of the exterior field $-V_{\nu_\alpha}$. Let R_n be the rational function whose numerator is of degree less than or equal to $n-1$ and with fixed poles at the zeros of w_n^α which interpolates p_m^* at the zeros of w_n^β. From Hermite's formula for the remainder of interpolation we have

$$p_m^*(x) - R_n(x) = \frac{1}{2\pi i} \int_\Gamma \frac{w_n^\beta(x)/w_n^\alpha(x)}{w_n^\beta(z)/w_n^\alpha(z)} \frac{p_m^*(z)}{z - x}\, dz, \qquad x \in [-1, 1],$$

where $\Gamma = \{z : g_{[-1,1]}(z, \infty) = \rho\}$ and ρ is sufficiently small so that the zeros of w_n^α and their accumulation points lie outside this curve. Therefore, considering (4.13), we obtain

$$|p_m^*(x) - R_n(x)| \leq C_8 \|F\|_\infty \frac{\sup_{x \in [-1-1]} |w_n^\beta(x)/w_n^\alpha(x)|}{\inf_{z \in \Gamma} |w_n^\beta(z)/w_n^\alpha(z)|} \exp\{m\rho\}. \tag{4.14}$$

For $\epsilon > 0$, since

$$\lim_n \left| \frac{w_n^\beta(z)}{w_n^\alpha(z)} \right|^{1/n} = \exp\{V_{\nu_\alpha - \tilde{\nu}\alpha}(z)\}$$

uniformly for $z \in \Gamma$, and

$$\limsup_{n} \left| \frac{w_n^{\beta}(x)}{w_n^{\alpha}(x)} \right|^{1/n} \le \exp\left\{ V_{\nu_\alpha - \tilde{\nu}\alpha}(x) \right\},$$

uniformly for $x \in [-1, 1]$, we obtain that for $n \ge n_0$

$$\left| \frac{w_n^{\beta}(z)}{w_n^{\alpha}(z)} \right| \ge \exp\left\{ n\left(V_{\nu_\alpha - \tilde{\nu}\alpha}(z) - \epsilon \right) \right\}, \qquad z \in \Gamma, \tag{4.15}$$

and

$$\begin{aligned} \left| \frac{w_n^{\beta}(x)}{w_n^{\alpha}(x)} \right| &\le \exp\left\{ n\left(V_{\nu_\alpha - \tilde{\nu}\alpha}(x) + \epsilon \right) \right\}, \qquad x \in [-1, 1] \\ &= \exp\{ n(C + \epsilon) \}, \end{aligned} \tag{4.16}$$

where C is the constant value that $V_{\nu_\alpha - \tilde{\nu}\alpha}$ takes on $[-1, 1]$. Using (4.14)–(4.16), we find that

$$| p_m^*(x) - R_n(x) | \le C_9 \|F\|_\infty \exp\left\{ m\rho + n\left[C + 2\epsilon - \inf_{z \in \Gamma} \left(V_{\nu_\alpha - \tilde{\nu}\alpha}(z) \right) \right] \right\}, \qquad x \in [-1, 1].$$

Take $m = m(n) = [n/l]$, where l is a properly chosen constant. Then $m(n) \le n/l$ and

$$| p_{m(n)}^*(x) - R_n(x) | \le C_9 \|F\|_\infty \exp\left\{ n\left[\frac{\rho}{l} + C + 2\epsilon - \inf_{z \in \Gamma} \left(V_{\nu_\alpha - \tilde{\nu}\alpha}(z) \right) \right] \right\}, \qquad x \in [-1, 1]. \tag{4.17}$$

It is known that ([9])

$$\inf_{z \in \Gamma} \left(V_{\nu_\alpha - \tilde{\nu}\alpha}(z) \right) > C.$$

Therefore, if we choose $\epsilon > 0$ sufficiently small and l sufficiently large and we fix them, we can assert that

$$\frac{\rho}{l} + C + 2\epsilon - \inf_{z \in \Gamma} \left(V_{\nu_\alpha - \tilde{\nu}\alpha}(z) \right) = -K, \tag{4.18}$$

where K is a constant greater than zero.

On the other hand,

$$\|F\|_\infty = C(F)\left(\max_{x \in [-1, 1]} F(x) - \min_{x \in [-1, 1]} F(x) \right) \le 2nC(F)\,\omega\left(F, \frac{1}{n} \right). \tag{4.19}$$

In fact, let $F(x_1) = \max_{x\in[-1,1]} F(x)$, $F(x_2) = \min_{x\in[-1,1]} F(x)$. Assume that $x_1 < x_2$. If $x_1 > x_2$, the proof is analogous. Then

$$\begin{aligned}
&\max_{x\in[-1,1]} F(x) - \min_{x\in[-1,1]} F(x) \\
&\quad = \left| \max_{x\in[-1,1]} F(x) - \min_{x\in[-1,1]} F(x) \right| \\
&\quad \le \left| \sum_{i=0}^{2n-1} \left[F\left(x_1 + \frac{i(x_2 - x_1)}{2n} \right) - F\left(x_1 + \frac{(i+1)(x_2 - x_1)}{2n} \right) \right] \right| \\
&\quad \le \sum_{i=0}^{2n-1} \left| F\left(x_1 + \frac{i(x_2 - x_1)}{2n} \right) - F\left(x_1 + \frac{(i+1)(x_2 - x_1)}{2n} \right) \right| \\
&\quad \le 2n\,\omega\left(F, \frac{1}{n} \right).
\end{aligned}$$

From (4.17)–(4.19) we obtain

$$\left| p^*_{m(n)}(x) - R_n(x) \right| \le C_{10} C(F)\,\omega\left(F, \frac{1}{n} \right) \exp\{\ln n - nK\}. \tag{4.20}$$

Finally, using Bernstein's Theorem and (4.20), we find that $(n \ge l)$

$$\begin{aligned}
&|F(x) - R_n(x)| \\
&\quad \le \left| F(x) - p^*_{m(n)}(x) \right| + \left| p^*_{m(n)}(x) - R_n(x) \right| \\
&\quad \le C_{11}\,\omega\left(F, \frac{1}{m(n)} \right) + C_{10} C(F)\,\omega\left(F, \frac{1}{n} \right) \exp\{\ln n - nK\} \\
&\quad = C_{11}\,\omega\left(F, \frac{1}{[n/l]} \right) + C_{10} C(F) \exp\{\ln n - nK\}\,\omega\left(F, \frac{1}{n} \right) \\
&\quad \le C_{11}\,\omega\left(F, \frac{2l}{n} \right) + C_{10} C(F) \exp\{\ln n - nK\}\,\omega\left(F, \frac{1}{n} \right) \\
&\quad \le \left[(2l+1) C_{11} + C_{10} C(F) \exp\{\ln n - nK\} \right] \omega\left(F, \frac{1}{n} \right),
\end{aligned}$$

which implies what we needed to prove. In these inequalities we used $[n/l] \ge n/2l$ if $n \ge l$; $\omega(F, \delta_1) \le \omega(F, \delta_2)$, for $\delta_1 \le \delta_2$; $\omega(F, \lambda\delta) \le (\lambda + 1)\omega(F, \delta)$; and $\lim_n \exp(\ln n - nK) = 0$. This concludes the proof of Theorem 4. ∎

Note that if F has a zero on $[-1,1]$ then $C(F) \leq 1$ and for this subclass of continuous functions we obtain in (4.12) an absolute constant.

In the case that we consider the best approximation of F by means of rational functions of the form P/w_n^α, $P \in \Pi_n$, it is easy to verify that in (4.12) the dependence on F of the constant in the right-hand side as well as the restriction that F be non-constant may be removed. Thus a direct extension of Bernstein's Theorem is obtained in this context.

COROLLARY 1. *If* $F \in \operatorname{Lip}(\lambda)$, $\lambda > \frac{1}{2}$, *then*

$$|I_n(F) - I(F)| = O\left(\frac{1}{n^{\lambda-1/2}}\right).$$

In particular,

$$\lim_{n\to\infty} I_n(F) = I(F).$$

Remark 4. Condition (3.4) is unnecessary for Theorem 4. In case of any table α compactly contained in $\overline{\mathbb{C}} \setminus [-1,1]$, the same result holds. The proof follows the same guidelines using the fact that the family of functions $\{n^{-1}\ln|w_n^\alpha|\}$ is uniformly bounded on each compact subset contained in the complement of the closure of the set of points in α. This allows us to reduce the proof to convergent subsequences for which the arguments of Theorem 4 hold. Apparently the constants which appear by multiplying by $\omega(F, 1/n)$ may depend on the convergent subsequence under consideration, but using the fact that α is bounded away from $[-1,1]$, an upper bound for such constants is not hard to achieve.

5. NUMERICAL EXAMPLES

With illustrative character and in order to check the effectiveness of the quadrature rules generated in the sections above, several numerical examples will be displayed in this section, where, for simplicity, we shall restrict our attention to formulas (1.3) not involving values of derivatives. In other words, we shall assume that the zeros of the polynomials w_n^β given by (2.2) satisfy $\beta_{n,j} \neq \beta_{n,k}$ if $j \neq k$, so that we have

$$I_n(F) = \sum_{j=1}^{n} A_j^n F(\beta_{n,j}). \tag{5.1}$$

For the sake of completeness we briefly recall how to obtain (5.1). Indeed, for a given triangular table $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ of points compactly contained in $\overline{\mathbb{C}} \setminus [-1,1]$ and a measure μ supported on $[-1,1]$, the "generalized moments”

$$\mu_{ij} = \int_{-1}^{+1} \frac{d\mu(x)}{(x - \alpha_{n,j})^i}, \qquad i = 1, \ldots, p_{n,j};\ j = 1, 2, \ldots, N;\ \sum_{j=1}^{N} p_{n,j} = n, \tag{5.2}$$

where $p_{n,j}$ denotes the multiplicity of $\alpha_{n,j}$, need to be computed. From (5.2), an MPTA to $\hat{\mu}(z)$ with denominator $w_n^{\beta}(z) = \prod_{j=1}^{n}(z - \beta_{n,j})$ can be constructed (details for the computations can be found, e.g., in [8]). Set

$$(n - 1/n)_{\hat{\mu}}(z) = \frac{Q_{n-1}(z)}{w_n^{\beta}(z)}.$$

Then, from Lemma 1, one has

$$A_j^n = \frac{Q_{n-1}(\beta_{n,j})}{(w_n^{\beta})'(\beta_{n,j})}, \qquad j = 1, 2, \ldots, n.$$

Therefore, the nodes and weights in (5.1) are explicitly determined. Next, several choices of tables α will be proposed. We start with simple Newtonian tables, as for example

$$\alpha = \{\alpha_n : n \in \mathbb{N}\},$$

where

$$\alpha_n = \{2, 3, \ldots, n + 1\} \qquad \text{or} \qquad \alpha_n = \{-2, -3, \ldots, -n - 1\}.$$

In both cases $\nu^{\alpha} = \delta_{\infty}$ and taking $w_n^{\beta}(z)$ as the nth orthogonal polynomial with respect to a positive finite Borel measure, say σ, on $[-1, 1]$, with $\sigma' > 0$ a.e. in $[-1, 1]$, represents an adequate selection. In all cases treated below, the quadrature arising when taking $d\sigma(x) = dx/\pi\sqrt{1 - x^2}$ (Chebyshev) or $d\sigma(x) = dx$ (Legendre) will be denoted by “Cheby” and “Legen”, respectively.

As for non-Newtonian tables, we shall consider the three following choices. Namely,

- $\alpha^{(1)} = \{\alpha_{n,j}^{(1)},\ j = 1, \ldots, n;\ n = 1, 2, \ldots\}$ where $1/\alpha_{n,j}^{(1)}$ are the zeros of $T_n(x)$, the nth Chebyshev polynomial of the first kind on $[-1, 1]$. This choice is included for a merely comparative purpose, because in this case α is not compactly contained in $\overline{\mathbb{C}} \setminus [-1, 1]$.

- $\alpha^{(2)} = \{\alpha_{n,j}^{(2)},\ j = 1, \ldots, n;\ n = 1, 2, \ldots\}$, $1/\alpha_{n,j}^{(2)}$ being the zeros of $T_n^{[-1/2, 1/2]}(x)$, the nth Chebyshev polynomial on $[-\frac{1}{2}, \frac{1}{2}]$.

• $\alpha^{(3)} = \{\alpha_{n,j}^{(3)},\ j = 1,\dots,n;\ n = 1,2,\dots\}$, $\alpha_{n,j}^{(3)}$ being the zeros of $T_n^{[-3,-2]}(x)$, the nth Chebyshev polynomial on $[-3,-2]$.

Corresponding to these choices of tables α, we have the following tables $\beta^{(k)}$, $k = 1,2,3$, defined as follows (see Remark 2):

• $\beta^{(k)} = \{\beta_{n,j}^{(k)},\ j = 1,\dots,n;\ n = 1,2,\dots; k = 1,2,3\}$, $\{\beta_{n,j}^{(k)}\}_{j=1}^{n}$ being the zeros of the nth orthogonal polynomial with respect to the varying measures

$$d\sigma_n(x) = \frac{dx}{\sqrt{1-x^2}\prod_{j=1}^{n}\left|x-\alpha_{n,j}^{(k)}\right|^2}, \qquad k = 1,2,3. \tag{5.3}$$

The quadrature formulas arising with these selections of points $\beta_{n,j}$ will be denoted by $\beta^{(1)}$, $\beta^{(2)}$, and $\beta^{(3)}$ in the tables displayed below. On the other hand, the examples considered deal with measures of the form $d\mu(x) = \omega(x)\,dx$ as indicated in Section 1.

Finally, points $\beta_{n,j}$ equally distributed on $(-1,1)$ are also considered. The corresponding quadrature will be denoted by "Equi".

In some examples, we have also computed estimations of the rate of convergence, i.e., an upper bound of

$$\lim_{n\to\infty}\left|I(F) - I_n(F)\right|^{1/n},$$

in order to compare it with $|I(F) - I_n(F)|^{1/n}$. Such entries appear in the columns labeled "Ratio". We distinguish two cases:

Case 1: $\nu^{\alpha} = \delta_\infty$. This happens for Newtonian choices, e.g.,

$$\alpha_n = \{2,3,\dots,n+1\}, \qquad n \in \mathbb{N}.$$

As indicated above, suitable $\{\beta_{n,j}\}_{j=1}^{n}$ could be zeros of "extremal polynomials" (see [4]) on $[-1,1]$, e.g., Chebyshev or Legendre polynomials.

Under these conditions, one has (see [4])

$$\lim_{n\to\infty}\left|I(F) - I_n(F)\right|^{1/n} \le \frac{1}{\tau}, \tag{5.4}$$

where $\tau = \sup\{\rho\colon F \in \mathscr{H}(D_\rho)\}$ and $D_\rho = \{z : |z + \sqrt{z^2-1}| < \rho\}$, and the branch in $\phi(z) = z + \sqrt{z^2-1}$ is taken so that $\phi(\infty) = \infty$. For example, if we take $F(x) = \sqrt{2}/(3+x)$, then $\tau = |-3-\sqrt{9-1}| = 3 + 2\sqrt{2}$ and consequently

$$\lim_{n\to\infty}\left|I(F) - I_n(F)\right|^{1/n} \le 0.17157\dots. \tag{5.5}$$

Case 2: $\nu_n^\alpha \to^*_{n\in\mathbb{N}} \nu^\alpha$ (general multipoint situation); ν^α being a probability measure supported on $\overline{\mathbb{C}} \setminus [-1,1]$. For simplicity, we shall only use tables α concentrated on $\mathbb{R} \setminus [-1,1]$.

Let us now consider Green's potential associated with ν^α, i.e.,

$$G^\alpha_{[-1,1]}(z) = \int g_{[-1,1]}(z,t)\, d\nu^\alpha(t),$$

where $g_K(z,t)$ denotes Green's function of $\overline{\mathbb{C}} \setminus K$ (K a compact) with pole at t.

By Theorem 2, we have now

$$\lim_{n\to\infty} |E_n(F)|^{1/n} = \lim_{n\to\infty} |I(F) - I_n(F)|^{1/n} \le \exp(-\tau),$$

with $\tau = \sup\{\rho : F \in \mathscr{H}(D_\rho)\}$ and $D_\rho = \{z : G^\alpha_{[-1,1]}(z) < \rho\}$.

In order to obtain $G^\alpha_{[-1,1]}(z)$ it should be taken into account that, as is well known,

$$g_{[-1,1]}(z;\infty) = \ln\left|z + \sqrt{z^2-1}\right|.$$

Now for $t \in \mathbb{R} \setminus [-1,1]$, let us consider the Moebius transformation h mapping $\overline{\mathbb{C}} \setminus [-1,1]$ onto itself, with $h(t) = \infty$, that is,

$$h(z) = \frac{1-tz}{z-t}$$

yielding

$$g_{[-1,1]}(z,t) = \ln\left|\left(\frac{1-tz}{z-t}\right) + \sqrt{\left(\frac{1-tz}{z-t}\right)^2 - 1}\right|. \tag{5.6}$$

Thus, as for the choice $\alpha^{(3)}$, we have (recall $w_n^\alpha(x) = T_n^{[-3,-2]}(x)$, where, as was already indicated, $T_n^{[a,b]}(x)$ denotes the nth Chebyshev polynomial of the first kind on the interval $[a,b]$)

$$d\nu^\alpha(x) = \frac{1}{\pi}\frac{dx}{\sqrt{(x+3)(-x-2)}}, \qquad -3 < x < -2. \tag{5.7}$$

On the other hand, the choice $\alpha^{(2)}$ gives ($w_n^\alpha(x) = x^n T_n^{[-1/2,1/2]}(x^{-1})$)

$$d\nu^\alpha(x) = \frac{2}{\pi}\frac{dx}{|x|\sqrt{x^2-4}}, \qquad |x| > 2. \tag{5.8}$$

From (5.6) and (5.7) we deduce

$$G_{[-1,1]}^{\alpha^{(3)}}(z) = \frac{1}{\pi}\int_{-3}^{-2} \ln\left|\left(\frac{1-tz}{z-t}\right) + \sqrt{\left(\frac{1-tz}{z-t}\right)^2 - 1}\right| \frac{dt}{\sqrt{-t^2-5t-6}} \tag{5.9}$$

and

$$G_{[-1,1]}^{\alpha^{(2)}}(z) = \frac{2}{\pi}\int_{I} \ln\left|\left(\frac{1-tz}{z-t}\right) + \sqrt{\left(\frac{1-tz}{z-t}\right)^2 - 1}\right| \frac{dt}{|t|\sqrt{t^2-4}}, \quad I = \mathbb{R} \setminus [-2,2]. \tag{5.10}$$

In the examples below, τ is estimated by (5.10) evaluating the function $G_{[-1,1]}^{\alpha}$ at the pole of the integrand F closest to $[-1,1]$, where the distance is measured in terms of the respective Green potentials. The tables below contain the absolute error of the quadrature formulas. The computations have been made on a HP workstation using quadruple precision. Some complementary calculations were performed using Maple V (Waterloo Maple Software).

EXAMPLE 1.

$$\omega(x) = \frac{1}{\sqrt{1+x}}.$$

For this measure, different integrands $F(x)$ will be considered.

- $F(x) = 1$

TABLE I

$\{\alpha_n\} = \{n : n = 2, 3, \dots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	3.1×10^{-02}	4.3×10^{-02}	1.0×10^{-01}	4.6×10^{-02}	3.1×10^{-02}	1.4×10^{-01}
4	3.1×10^{-04}	2.3×10^{-04}	8.7×10^{-04}	2.4×10^{-04}	7.9×10^{-04}	3.3×10^{-03}
6	1.1×10^{-06}	7.8×10^{-07}	2.2×10^{-06}	6.4×10^{-07}	1.6×10^{-05}	4.0×10^{-05}
8	2.6×10^{-09}	1.8×10^{-09}	1.8×10^{-09}	1.5×10^{-09}	1.5×10^{-07}	3.0×10^{-07}
10	4.3×10^{-12}	3.1×10^{-12}	2.3×10^{-12}	2.4×10^{-12}	6.5×10^{-10}	1.5×10^{-09}

TABLE II

$\{\alpha_n\} = \{n : n = -2, -3, \dots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	7.5×10^{-02}	2.1×10^{-01}	5.9×10^{-02}	4.8×10^{-02}	3.5×10^{-02}	3.9×10^{-01}
4	2.3×10^{-04}	3.2×10^{-03}	8.0×10^{-05}	1.5×10^{-05}	2.3×10^{-04}	1.6×10^{-02}
6	2.2×10^{-07}	2.2×10^{-05}	2.9×10^{-06}	3.4×10^{-08}	9.7×10^{-07}	2.7×10^{-04}
8	7.0×10^{-11}	8.4×10^{-08}	1.1×10^{-08}	2.0×10^{-11}	3.1×10^{-09}	2.6×10^{-06}
10	2.5×10^{-14}	2.1×10^{-10}	2.3×10^{-11}	3.4×10^{-14}	7.8×10^{-12}	1.5×10^{-08}

TABLE III

$\{\alpha_n\} = \alpha^{(1)}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	1.0×10^{-01}	4.3×10^{-01}	2.3×10^{-01}	3.5×10^{-02}	1.1×10^{-02}	8.6×10^{-01}
4	2.0×10^{-02}	7.9×10^{-02}	3.4×10^{-03}	1.4×10^{-02}	3.6×10^{-03}	3.2×10^{-01}
6	3.8×10^{-03}	1.6×10^{-02}	4.4×10^{-05}	2.6×10^{-03}	1.3×10^{-03}	1.4×10^{-01}
8	7.5×10^{-04}	3.5×10^{-03}	1.4×10^{-07}	4.9×10^{-04}	4.8×10^{-04}	6.4×10^{-02}
10	1.6×10^{-04}	7.9×10^{-04}	2.5×10^{-08}	9.4×10^{-05}	1.9×10^{-04}	3.1×10^{-02}

TABLE IV

$\{\alpha_n\} = \alpha^{(2)}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	6.4×10^{-03}	5.6×10^{-02}	6.9×10^{-02}	1.9×10^{-02}	1.3×10^{-02}	1.4×10^{-01}
4	5.3×10^{-05}	7.6×10^{-04}	8.0×10^{-04}	3.5×10^{-05}	5.4×10^{-04}	4.7×10^{-03}
6	5.8×10^{-07}	1.1×10^{-05}	7.2×10^{-06}	3.1×10^{-07}	1.4×10^{-05}	1.7×10^{-04}
8	4.9×10^{-09}	1.8×10^{-07}	6.6×10^{-08}	2.9×10^{-09}	3.4×10^{-07}	6.3×10^{-06}
10	5.6×10^{-11}	2.9×10^{-09}	6.0×10^{-10}	3.2×10^{-11}	8.9×10^{-09}	2.3×10^{-07}

TABLE V

$\{\alpha_n\} = \alpha^{(3)}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	6.8×10^{-02}	2.0×10^{-01}	6.0×10^{-02}	4.2×10^{-02}	3.5×10^{-02}	3.7×10^{-01}
4	2.0×10^{-03}	1.7×10^{-02}	1.7×10^{-03}	4.8×10^{-04}	5.3×10^{-04}	8.1×10^{-02}
6	6.8×10^{-05}	1.7×10^{-03}	2.8×10^{-04}	7.6×10^{-06}	9.5×10^{-06}	1.9×10^{-02}
8	2.4×10^{-06}	1.7×10^{-04}	9.1×10^{-06}	1.3×10^{-07}	2.2×10^{-07}	4.7×10^{-03}
10	8.8×10^{-08}	1.8×10^{-05}	7.3×10^{-08}	2.3×10^{-09}	6.3×10^{-09}	1.2×10^{-03}

- $F(x) = 8\sqrt{2}/(16 + (x + 1)^4)$.

TABLE VI

$\{\alpha_n\} = \{n : n = 2, 3, \ldots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	9.6×10^{-03}	3.4×10^{-02}	6.0×10^{-02}	1.9×10^{-02}	4.6×10^{-02}	8.0×10^{-02}
4	8.3×10^{-04}	1.1×10^{-03}	4.5×10^{-03}	1.2×10^{-03}	6.1×10^{-03}	7.3×10^{-03}
6	7.1×10^{-05}	6.9×10^{-05}	3.1×10^{-05}	3.1×10^{-05}	1.2×10^{-03}	4.5×10^{-04}
8	4.9×10^{-06}	7.4×10^{-07}	1.9×10^{-05}	2.8×10^{-06}	3.6×10^{-05}	2.7×10^{-04}
10	6.5×10^{-08}	3.2×10^{-07}	1.6×10^{-06}	3.9×10^{-08}	6.5×10^{-05}	5.4×10^{-05}

TABLE VII

$\{\alpha_n\} = \{n : n = -2, -3, \ldots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	3.6×10^{-02}	4.3×10^{-02}	1.2×10^{-01}	5.2×10^{-02}	1.3×10^{-02}	1.4×10^{-01}
4	2.2×10^{-03}	3.6×10^{-02}	4.1×10^{-03}	8.4×10^{-04}	1.3×10^{-03}	2.2×10^{-01}
6	4.0×10^{-05}	6.0×10^{-03}	4.8×10^{-04}	3.4×10^{-06}	2.7×10^{-04}	9.3×10^{-02}
8	2.0×10^{-07}	4.1×10^{-04}	1.3×10^{-05}	2.6×10^{-07}	2.0×10^{-05}	1.3×10^{-02}
10	2.8×10^{-08}	1.2×10^{-05}	3.3×10^{-06}	1.6×10^{-08}	3.7×10^{-07}	2.7×10^{-03}

- $F(x) = 2\sqrt{2}/(4 + (x+1)^2)$.

TABLE VIII

$\{\alpha_n\} = \{n : n = -2, -3, \ldots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	1.9×10^{-02}	9.2×10^{-02}	4.8×10^{-02}	4.8×10^{-03}	2.5×10^{-02}	2.0×10^{-01}
4	6.3×10^{-05}	2.7×10^{-03}	9.2×10^{-04}	1.7×10^{-04}	4.0×10^{-04}	1.9×10^{-02}
6	8.4×10^{-08}	1.7×10^{-04}	8.3×10^{-06}	4.4×10^{-07}	9.8×10^{-06}	2.8×10^{-03}
8	1.5×10^{-09}	1.2×10^{-05}	1.5×10^{-06}	2.0×10^{-09}	4.2×10^{-07}	4.4×10^{-04}
10	4.7×10^{-11}	6.6×10^{-07}	7.7×10^{-08}	9.0×10^{-11}	2.4×10^{-08}	55.6×10^{-05}

- $F(x) = \sqrt{2}/(x+3)$.

TABLE IX

$\{\alpha_n\} = \alpha^{(3)}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	9.1×10^{-04}	2.0×10^{-03}	2.3×10^{-04}	6.9×10^{-04}	2.7×10^{-04}	3.4×10^{-03}
4	2.0×10^{-07}	1.3×10^{-06}	2.1×10^{-07}	6.9×10^{-08}	2.5×10^{-08}	5.6×10^{-06}
6	4.8×10^{-11}	9.4×10^{-10}	1.5×10^{-10}	8.0×10^{-12}	3.9×10^{-12}	9.8×10^{-09}
8	1.2×10^{-14}	7.0×10^{-13}	2.7×10^{-14}	9.4×10^{-16}	8.8×10^{-16}	1.7×10^{-11}
10	5.8×10^{-17}	4.8×10^{-16}	6.7×10^{-17}	6.1×10^{-17}	6.1×10^{-17}	3.1×10^{-14}

- $F(x) = (\sqrt{2}/2)(1/(1 + e^{((x+1)/2)}))$.

TABLE X

$\{\alpha_n\} = \{n : n = -2, -3, \ldots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	2.9×10^{-04}	1.1×10^{-02}	9.8×10^{-03}	1.8×10^{-03}	3.0×10^{-03}	2.6×10^{-02}
4	9.3×10^{-05}	1.2×10^{-03}	2.1×10^{-05}	4.8×10^{-06}	7.5×10^{-05}	6.0×10^{-03}
6	1.0×10^{-07}	1.3×10^{-05}	1.4×10^{-06}	1.6×10^{-08}	6.6×10^{-07}	1.7×10^{-04}
8	1.1×10^{-10}	6.1×10^{-08}	1.0×10^{-08}	1.0×10^{-11}	1.7×10^{-09}	1.8×10^{-06}
10	9.2×10^{-14}	1.5×10^{-09}	1.6×10^{-10}	1.9×10^{-13}	5.4×10^{-11}	1.1×10^{-07}

- $F(x) = (\sqrt{2}/4)(x+1)/e^{((x-1)/2)}$.

TABLE XI
$\{\alpha_n\} = \{n : n = -2, -3, \ldots\}$

n	Cheby	Legen	$\beta^{(1)}$	$\beta^{(2)}$	$\beta^{(3)}$	Equi
2	1.1×10^{-01}	2.5×10^{-01}	2.1×10^{-02}	8.6×10^{-02}	3.7×10^{-02}	4.3×10^{-01}
4	5.9×10^{-04}	8.6×10^{-03}	2.3×10^{-04}	6.3×10^{-05}	6.3×10^{-04}	4.3×10^{-02}
6	6.8×10^{-08}	9.3×10^{-06}	9.5×10^{-07}	2.2×10^{-09}	9.2×10^{-07}	1.4×10^{-04}
8	5.3×10^{-10}	5.2×10^{-07}	7.4×10^{-08}	8.6×10^{-11}	1.9×10^{-08}	1.6×10^{-05}
10	2.5×10^{-13}	1.7×10^{-09}	1.9×10^{-10}	3.0×10^{-13}	6.5×10^{-11}	1.3×10^{-07}

- $F(x) = |x|$.

TABLE XII
$\alpha = \alpha^{(2)}$

n	$\beta^{(2)}$
2	3.3×10^{-01}
4	6.1×10^{-02}
6	2.7×10^{-02}
8	1.5×10^{-02}
10	9.5×10^{-03}

TABLE XIII
$\alpha = \alpha^{(3)}$

n	$\beta^{(3)}$
2	1.7×10^{-01}
4	1.0×10^{-01}
6	1.6×10^{-02}
8	1.5×10^{-02}
10	4.4×10^{-03}

TABLE XIV

n	Ratio
2	2.2×10^{-01}
4	1.4×10^{-01}
6	1.8×10^{-01}
8	2.0×10^{-01}
10	2.1×10^{-01}

TABLE XV

n	Ratio
2	1.1×10^{-01}
4	2.2×10^{-01}
6	2.5×10^{-01}
8	2.4×10^{-01}
10	2.6×10^{-01}

TABLE XVI

n	Ratio
2	1.6×10^{-01}
4	1.0×10^{-01}
6	1.4×10^{-01}
8	1.4×10^{-01}
10	1.5×10^{-01}

In the latter example, the integrand function $F(x)$ is not analytic. Since $F(x) = |x|$ clearly satisfies a Lipschitz condition with $\lambda = 1$, then by Corollary 1 convergence holds. However, the rate is now slower than for analytic integrands, as can be deduced from Tables XII and XIII.

Next, and for the same measure $d\mu(x) = (1+x)^{-1/2}\,dx$, estimations of the rate of convergence will be given. Thus, for $F(x) = 2\sqrt{2}/(16 + (x+1)^4)$, it results that $\exp(-\tau) = 0.35$ (Table XIV) for $\alpha = \alpha^{(2)}$, $\beta = \beta^{(2)}$, and Table XV for $\alpha = \alpha^{(3)}$, $\beta = \beta^{(3)}$). Table XVI collects the estimations for $\alpha = \alpha^{(3)}$, $\beta = \beta^{(3)}$, and $F(x) = 2\sqrt{2}/(4 + (x+1)^2)$, for which $\exp(-\tau) = 0.24$.

TABLE XVII
$\{\alpha_n\} = \{n : = 2, , \ldots\}$

n	Chebyshev		Legendre	
	Error	Ratio	Error	Ratio
2	2.9×10^{-02}	1.7×10^{-01}	6.1×10^{-02}	2.5×10^{-01}
4	2.2×10^{-03}	2.2×10^{-01}	8.7×10^{-03}	3.1×10^{-01}
6	1.0×10^{-04}	2.2×10^{-01}	5.5×10^{-04}	2.9×10^{-01}
8	5.4×10^{-07}	1.6×10^{-01}	6.5×10^{-06}	2.2×10^{-01}
10	3.5×10^{-07}	2.3×10^{-01}	2.8×10^{-06}	2.8×10^{-01}
12	2.1×10^{-08}	2.3×10^{-01}	1.9×10^{-07}	2.7×10^{-01}

EXAMPLE 2.

$$\omega(x) = -\ln\left(\frac{x+1}{2}\right)\bigg/\sqrt{1+x}\,.$$

- $F(x) = 8\sqrt{2}\,/(16 + (x+1)^4)$.With this function F, and α as indicated in Table XVII one now has $\exp(-\tau) = 0.22$.
- $F(x) = 2\sqrt{2}\,/(4 + (x+1)^2)$. Now, $\exp(-\tau) = 0.31$ (Table XVIII).

EXAMPLE 3.

$$\omega(x) = (1 - x^2)^{1+i}$$

In this case, we have a complex Jacobi weight function, i.e., $\omega(x) = (1-x)^a(1+x)^b$, with $\mathrm{Re}(a) > -1$ and $\mathrm{Re}(b) > -1$, corresponding to $a = b = 1 + i$ (see [2] and [4] concerning the polynomial situation).

- $F(x) = \exp(x)$.Since $F(x)$ is an entire function, obviously $\exp(-\tau)$

TABLE XVIII
$\{\alpha_n\} = \{n : n = 2, 3, \ldots\}$

n	Chebyshev		Legendre	
	Error	Ratio	Error	Ratio
2	2.6×10^{-02}	1.6×10^{-01}	1.3×10^{-01}	3.5×10^{-01}
4	4.9×10^{-03}	2.6×10^{-01}	1.8×10^{-02}	3.7×10^{-01}
6	2.5×10^{-05}	1.7×10^{-01}	1.3×10^{-04}	2.2×10^{-01}
8	6.5×10^{-06}	2.2×10^{-01}	4.1×10^{-05}	2.8×10^{-01}
10	1.3×10^{-07}	2.0×10^{-01}	9.6×10^{-07}	2.5×10^{-01}
12	5.8×10^{-09}	2.1×10^{-01}	5.1×10^{-08}	2.5×10^{-01}

TABLE XIX

$\{\alpha_n\} = \{n : n = 2, 3, \ldots\}$

	Chebyshev		Legendre	
n	Error	Ratio	Error	Ratio
2	7.9×10^{-02}	2.8×10^{-01}	3.9×10^{-02}	2.0×10^{-01}
4	6.5×10^{-04}	1.6×10^{-01}	4.7×10^{-04}	1.5×10^{-01}
6	3.8×10^{-06}	1.2×10^{-01}	2.6×10^{-06}	1.2×10^{-01}
8	9.9×10^{-09}	1.0×10^{-01}	7.7×10^{-09}	9.7×10^{-02}
10	8.4×10^{-11}	9.8×10^{-02}	9.7×10^{-11}	1.0×10^{-01}

TABLE XX

$\{\alpha_n\} = \{n : n = 2, 3, \ldots\}$

n	Error	Ratio
2	3.9×10^{-02}	2.0×10^{-01}
4	1.4×10^{-03}	1.9×10^{-01}
6	7.0×10^{-05}	2.0×10^{-01}
8	4.0×10^{-06}	2.1×10^{-01}

$= 0$, which can be checked from Table XIX.

- $F(x) = 2\sqrt{2}/(4 + (x+1)^2)$. Here $\exp(-\tau) = 0.2168$ (Table XX).

EXAMPLE 4. This example is a particular case of certain integrals of the form

$$\int_{-1}^{1} \omega \exp(-\omega(x+1))\, dx, \qquad \omega > 0,$$

studied by Van Assche *et al.* in a recent paper [13] and also concerning quadrature formulas which integrate exactly rational functions with prescribed poles. They propose the Newtonian table $\alpha = \{\alpha_j = -(1 - j^{-1/2)})^{-1},\ j = 1, \ldots\}$, whose choice is motivated by the fact that the integrand has more mass near -1 than near $+1$, especially for large ω. In this case a rather simple selection as $\alpha = \{n : n = -2, -3, \ldots\}$ in our method gives similar results for $\omega = 5$ (Table XXI) compared to those given in [13]. Furthermore, both rational quadrature rules give better results than the Gauss–Legendre quadrature rule. Actually, in [13], relative errors instead of absolute errors are handled. However, for $\omega = 5$, the value of the integral is very close to one. Thus, comparison can be performed.

With a general character, and as can be observed from the tables above,

TABLE XXI

n	Legendre	
	Error	Ratio
2	3.6×10^{-01}	6.0×10^{-01}
4	3.4×10^{-03}	2.4×10^{-01}
6	3.5×10^{-05}	1.8×10^{-01}
8	1.8×10^{-07}	1.4×10^{-01}
10	6.3×10^{-11}	9.5×10^{-02}
12	8.7×10^{-13}	9.9×10^{-02}
14	4.2×10^{-16}	8.0×10^{-02}
16	7.8×10^{-18}	8.5×10^{-02}

in the case of a Newtonian table α, the zeros of Chebyshev polynomials $T_n(x)$ produce, as a rule, the most acceptable results. On the other hand, the effectiveness of those choices proposed in Remark 2 for a non-Newtonian table α seems to be also confirmed through several numerical experiments. Even the extremely simple selection of nodes $\beta_{n,j}$ equidistant on $(-1, 1)$ yields, in most of the cases, rather good results. However, it should be said that for such a selection convergence is not guaranteed, at least, from a theoretical point of view.

Finally, it is convenient to point out that in both [13] and [14] quadrature formulas based on rational functions are considered. Here, the choice of the poles of the rational approximating functions is made taking into account the singularities of the integrand. Therefore, an alternative approach for constructing quadrature formuals exact for rational functions is presented in this paper.

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